Addition invariance with respect to bi-partitions, nullified players and proportional rules

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Cooperative game theory aims at combining axioms to study rules that distribute the worth of the grand coalition.

Two main types of rules:

- **marginalist (contributory) rules**, such as the Shapley value and the Banzhaf value.
- **egalitarian rules**, such as the equal division and the equal surplus division.

The two types can be compared axiomatically (van den Brink, 2007, Kamijo and Kongoc, Béal et al., 2012b).

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Weighted rules have been introduced to take exogenous asymmetries among the players into account.

Example: Kalai and Samet (1987) study weighted Shapley values.

We analyze weighted efficient distribution of the worth of the grand coalition, called Proportional rules.

Weights are exogenously given as in Kalai and Samet.

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We characterize the Proportional rules and the equal division rule by means of classical and new axioms. Two features:

1. We impose/formulate axioms exhibiting variations on the **null player** and **nullifying player** properties.
   
   **Example:** null player in a productive environment (Casajus and Huetttner, 2013).

2. We often impose an **axiom of invariance**, describing which modifications of a game preserve the payoffs recommended by a rule.

   **Examples:** Independence of irrelevant alternatives (Nash, 1953), Null player out property (Derks and Haller, 1999), Marginality (Young, 1985). See also (Béal et. al., 2012b).
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Definitions

\( N = \{1, \ldots, n\} \) is a **fixed and finite player set**.

\( v \in \{f : 2^N \rightarrow \mathbb{R}, f(\emptyset) = 0\} \) is a **(TU-)game** on \( N \).

The **dual game** of \( v \) is the game \( v^D \) such that

\[ v^D(S) = v(N) - v(N \setminus S), \quad \forall S \in 2^N. \]

In game \( v \), a player \( i \in N \) is called

- **null** if \( v(S) = v(S \setminus i) \) for all \( S \ni i \).
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Two players \( i, j \in N \) are **symmetric** in \( v \) if: \( v(S \cup i) = v(S \cup j) \), \( \forall S \subseteq N \setminus \{i, j\} \).
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A rule is a function $\varphi$ that assigns a payoff vector $\varphi(v)$ to all $v$.

The proportional rules (abbreviated $P$-rules) are given by

$$\forall v, \forall i \in N, \quad P_i(v) = \omega_i \cdot v(N).$$

for some (exogenously given) constants $\omega_i \in \mathbb{R}$, $\sum_{i \in N} \omega_i = 1$.

The $P^0$-rules are the $P$-rules such that $\omega_i \geq 0$, $\forall i \in N$.

The equal division rule $ED$ is the $P$-rule such that $\omega_i = 1/n$, $\forall i \in N$, i.e.

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- **Efficiency** if, $\forall v, \sum_{i \in N} \varphi_i(v) = v(N)$.

- **Symmetry** if, $\forall v, \forall i, j \in N$ symmetric, $\varphi_i(v) = \varphi_j(v)$.

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- **Additivity** if, $\forall v, w, \varphi(v + w) = \varphi(v) + \varphi(w)$.

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Variations on the null player property

- **Null player in a productive environment:**
  \[ \forall v \text{ with } v(N) \geq 0, \forall i \in N \text{ null, } \varphi_i(v) \geq 0. \]

Introduced by Casajus and Huetttner (2013).

For a game \( v \) and a player \( i \in N \), the game in which \( i \) is nullified is the game \( v^N_i \) given by

\[ \forall S \in 2^N, \quad v^N_i(S) = v(S \setminus \{i\}). \]

- **Nullified player:**
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Same flavor as population solidarity in Chun and Park (2013) and solidarity principle in Thomson (2012).
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Variation on the nullifying player property

- **Positive player:**
  \[ \forall v, \forall i \in N \text{ positive, } \varphi_i(v) \geq 0. \]

  \[ \forall a \in \mathbb{R}, \forall b \in \mathbb{R}^N, \text{ the game } (a \cdot v + b) \text{ is given by } \]
  \[ \forall S \in 2^N, \quad (a \cdot v + b)(S) = a \cdot v(S) + \sum_{i \in S} b_i. \]

- **Weak covariance:**
  \[ \forall i, j \in N, \text{ if } b_i = b_j, \text{ then } \varphi(a \cdot v + b) = a \cdot \varphi(v) + b. \]
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Addition invariance with respect to bi-partitions

∀v, ∀S \notin \{\emptyset, N\}, ∀c \in \mathbb{R}, the game v_{S,c} is given by

\forall T \in 2^N, \quad v_{S,c}(T) = \begin{cases} v(T) + c & \text{if } T \in \{S, N\setminus S\}, \\ v(T) & \text{otherwise.} \end{cases}

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Bi-partitions appear in the Shapley and Banzhaf values.

S and N\setminus S are somehow competing against each other in a two-entity bargaining.

v(S) and v(N\setminus S) can be seen as their bargaining powers.
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This aspect is highlighted for the Shapley value by Eisenman (1967), Rothblum (1988), Evans (1996) and Béal et al. (2012a).

Since in $\nu_{S,c}$, the worths of $S$ and $N \setminus S$ are reevaluated by the same amount, *ceteris paribus*, then the bargaining powers of $S$ and $N \setminus S$ are affected similarly.

Satisfied by the Shapley value, Banzhaf value, equal division rule ...
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The combination of self-duality and additivity implies addition invariance with respect to bi-partitions. Addition invariance with respect to bi-partitions implies self-duality.
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Characterizations

Efficiency
Symmetry
Null player
Nullifying player
Additivity
Add. Inv. w.r.t. bi-partitions
Null player in productive env.
Positive player
Nullified player
Weak covariance
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Punctual axioms

(Singh, 1962)
(Not P rules)
(P rules)
(Van den Brink, 2007)
(Béal et al., 2013)
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Shapley value (Shubik, 1962)

Equal division rule

Not P0-rules

P0-rules

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